Score Sequence Pair Problem of Bipartite Tournaments

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SUMMARY

For two nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, a pair of nonnegative integer sequences $S=\{S_1, S_2\}$ is a *score sequence pair of a bipartite k-tournament* if, for some positive integer k, there is a bipartite directed graph with two independent vertex sets $V_1=\{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2=\{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ such that $deg^+(v_{1j})=s_{1j}$, $deg^-(v_{1j})=k \cdot n_2-s_{1j}$, $deg^+(v_{2q})=s_{2q}$ and $deg^-(v_{2q})=k \cdot n_1-s_{2q}$ for each $j=1, 2, \cdots, n_1$, $q=1, 2, \cdots, n_2$. The *score sequence pair problem of a bipartite k-tournament* is: Given some positive integer k and a pair of nonnegative integer sequences, determine whether it is a score sequence pair of a bipartite k-tournament or not. In this paper, we consider the score sequence pair problem of a bipartite tournament and of a bipartite k-tournament, and give efficient algorithms.

Key words : *bipartite tournament, bipartite k-tournament and score sequence pair*

1. Introduction

Let T be a bipartite directed graph. T is a bipartite k-tournament if and only if, for some positive integer k and some nonnegative integer $k' \leq k$, T has k' edges (u, v) ((u, v) denotes the edge from u to v) and $k-k'$ edges (v, u) for any vertex pair $u, v \in T$. For two nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, a pair of nonnegative integer sequences $S=\{S_1, S_2\}$ is a *score sequence pair of a bipartite k-tournament* if there is a bipartite k-tournament with two independent vertex sets $V_1=\{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2=\{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ such that $deg^+(v_{1j})=s_{1j}$, $deg^-(v_{1j})=k \cdot n_2-s_{1j}$, $deg^+(v_{2q})=s_{2q}$ and $deg^-(v_{2q})=k \cdot n_1-s_{2q}$ for each $j=1, 2, \cdots, n_1$, $q=1, 2, \cdots, n_2$ ($deg^+(u)$ and $deg^-(u)$ are the outdegree and the indegree of u respectively). The *score sequence pair problem of a bipartite k-tournament* is: Given some positive integer k and a pair of nonnegative integer sequences $S=\{S_1, S_2\}$ with $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, determine whether S is a score sequence pair of a bipartite k-tournament or not. The score sequence of a tournament was considered by Landau[10]. The graphical degree sequence problems and the variations of them have been considered by Havel[9], Erdös and Gallai[4], Hakimi[6], Menon[11], Takahashi, Imai and Asano [1, 12] and others[2, 3, 8].

In this paper, we consider the score sequence problem of a bipartite tournament and of a bipartite k-tournament, and give efficient algorithms.

2. Score Sequence Pair Problem of a Bipartite Tournament

For two nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, a

Received Oct. 30, 1995

pair of nonnegative integer sequences $S=\{S_1, S_2\}$ is a *score sequence pair of a bipartite tournament* (*bitournament*, for short) if and only if there is a bitournament with two independent vertex sets $V_1=\{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2=\{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ such that $\deg^+(v_{1j})=s_{1j}$, $\deg^-(v_{1j})=n_2-s_{1j}$, $\deg^+(v_{2q})=s_{2q}$ and $\deg^-(v_{2q})=n_1-s_{2q}$ for each $j=1, 2, \cdots, n_1$, $q=1, 2, \cdots, n_2$. The *score sequence pair problem of a bitournament* is: Given a pair of nonnegative integer sequences $S=\{S_1, S_2\}$ with $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, determine whether S is a score sequence pair of a bitournament or not. We can assume without loss of generality that $0\leqq s_{11}\leqq s_{12}\leqq \cdots \leqq s_{1n_1}\leqq n_2$ and $0\leqq s_{21}\leqq s_{22}\leqq \cdots \leqq s_{2n_2}\leqq n_1$.

In this section, we consider the score sequence pair problem of a bitournament and present efficient algorithms. We first present an algorithm to determine whether $S=\{S_1, S_2\}$ is a score sequence pair of a bitournament or not in $O(n)$ time, where $n=n_1+n_2$. We can obtain the following proposition.

Proposition 2.1. Let $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$ be two nonnegative integer sequences, where $0\leqq s_{11}\leqq s_{12}\leqq \cdots \leqq s_{1n_1}\leqq n_2$ and $0\leqq s_{21}\leqq s_{22}\leqq \cdots \leqq s_{2n_2}\leqq n_1$. Then $S=\{S_1, S_2\}$ is a score sequence pair of a bitournament if and only if

$$\sum_{j=1}^{t_1} s_{1j}+\sum_{j=1}^{t_2} s_{2j}\geqq t_1\cdot t_2$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$.

Proof. Let $B=(b_1, b_2, \cdots, b_{n_1})$ be a sequence of nonnegative integers with $b_j=n_2-s_{1j}$ for each $j=1, 2, \cdots, n_1$. Then it is clear that S is a score sequence pair of a bitournament if and only if $S'=\{B, S_2\}$ is bipartite graphical. Since $n_2 \geqq b_1\leqq b_2\geqq \cdots \geqq b_{n_1}\geqq 0$, S' is bipartite graphical if and only if

$$\sum_{j=1}^{t_1} b_j\leqq \sum_{j=1}^{t_2} s_{2j}+t_1(n_2-t_2)$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$[12]. Furthermore,

$$\sum_{j=1}^{t_1} b_j\leqq \sum_{j=1}^{t_2} s_{2j}+t_1(n_2-t_2)$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$, if and only if

$$\sum_{j=1}^{t_1} s_{1j}+\sum_{j=1}^{t_2} s_{2j}\geqq t_1\cdot t_2$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$.

By the argument above, the proof of this proposition is completed. ■

Based on Proposition 2.1, we can determine whether $S=\{S_1, S_2\}$ is a score sequence pair of a bitournament or not in $O(n)$ time as follows.

For each $t_1=1, 2, \cdots, n_1$, we consider $z(t_1)$ defined by $z(t_1)=\max\{j \mid s_{2j}<t_1\}$. Then Proposition 2.1 can be rewritten as follows: S is a score sequence pair of a bitournament if and only if

$$\sum_{j=1}^{t_1} s_{1j}+\sum_{j=1}^{z(t_1)} s_{2j}\geqq t_1\cdot z(t_1)$$

for each $t_1=1, 2, \cdots, n_1$, since

$$\sum_{j=1}^{t_1} s_{1j}+\sum_{j=1}^{z(t_1)} s_{2j}\geqq t_1\cdot z(t_1)$$

if and only if

$$\sum_{j=1}^{t_1} b_j+\sum_{j=1}^{z(t_1)} s_{2j}+t_1(n_2-z(t_1))$$

and

$$\sum_{j=1}^{z(t_1)} s_{2j}+t_1(n_2-z(t_1))\leqq \sum_{j=1}^{t_2} s_{2j}+t_1(n_2-t_2)$$

for each $t_2=1, 2, \cdots, n_2$[12] (we assume $b_j=n_2-s_{1j}$ for each $j=1, 2, \cdots, n_1$). We can compute all $z(t_1)$ in $O(n)$ time as follows.

Algorithm Comp-$z(t_1)$.

Step1: $s_{20}:=0$, $s_{2,n_2+1}:=n_1$ and $q:=1$.

Step2: **For** $j:=0$ **to** n_2+1 **do** the following (a) and (b).

(a) **while** $s_{2j}\geqq q$ **do** $q:=q+1$ and $z(q):=z(q-1)$.

(b) $z(q):=j$.

Thus we can determine whether S is a score sequence pair of a bitournament or not in $O(n)$ time.

Next we present an algorithm for actually constructing a bitournament based on the following proposition, which can be proved easily.

Proposition 2.2. Let $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$ be two nonnegative integer sequences, where $0\leqq s_{11}\leqq s_{12}\leqq \cdots \leqq s_{1n_1}\leqq$

n_2 and $0 \leqq s_{21} \leqq s_{22} \leqq \cdots \leqq s_{2n_2} \leqq n_1$. Let $s_{1,n_1+1} = n_2 + 1$ and let $U = (u_1, u_2, \cdots, u_{n_1})$ be defined by using $t = s_{2n_2} + 1$, $x = \min\{j \mid s_{1j} = s_{1t}\}$ and $y = \max\{j \mid s_{1j} = s_{1t}\}$ as follows.

$$u_j = \begin{cases} s_{1j} - 1 & \text{if } x \leqq j \leqq y - t + x \text{ or } y + 1 \leqq j \leqq n_1, \\ s_{1j} & \text{if } 1 \leqq j \leqq x - 1 \text{ or } y - t + x + 1 \leqq j \leqq y. \end{cases}$$

Then $S = \{S_1, S_2\}$ is a score sequence pair of a bitournament if and only if $\{U, S_2 - s_{2n_2}\}$ is a score sequence pair of a bitournament, where $S_2 - s_{2n_2} = (s_{21}, s_{22}, \cdots, s_{2,n_2-1})$. Furthermore, $u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1} \leqq n_2 - 1$.

Proof. It is clear from the definition of U that $u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1} \leqq n_2 - 1$. The sufficiency is almost trivial. If $\{U, S_2 - s_{2n_2}\}$ is a score sequence pair and H is a bitournament with $\{U, S_2 - s_{2n_2}\}$ as a score sequence pair, then the bitournament T obtained from H by adding edges (v_{1j}, v_{2n_2}) for all j with $u_j = s_{1j} - 1$ and adding edges (v_{2n_2}, v_{1j}) for all j with $u_j = s_{1j}$ has S as a score sequence pair.

The necessity can be obtained as follows. Let T be a bitournament with S as a score sequence pair. If T contains edges (v_{1j}, v_{2n_2}) for all j with $u_j = s_{1j} - 1$ and edges (v_{2n_2}, v_{1j}) for all j with $u_j = s_{1j}$, then the bitournament H obtained from T by deleting the edges has $\{U, S_2 - s_{2n_2}\}$ as a score sequence pair. Hence T is assumed to contains no edge (v_{1t}, v_{2n_2}) for some t with $u_t = s_{1t} - 1$. Then T contains an edge (v_{1q}, v_{2n_2}) for some q with $u_q = s_{1q}$. Furthermore, since $s_{1t} = \deg^+(v_{1t}) \geqq s_{1q} = \deg^+(v_{1q})$, there is a vertex v_{2r} such that (v_{1t}, v_{2r}) is an edge of T but (v_{1q}, v_{2r}) is not in T. Thus T has a directed cycle $C = (v_{2n_2}, v_{1t}), (v_{1t}, v_{2r}), (v_{2r}, v_{1q}), (v_{1q}, v_{2n_2})$, since T is a bitournament. Let $E_1 = \{(v_{2n_2}, v_{1q}), (v_{1q}, v_{2r}), (v_{2r}, v_{1t}), (v_{1t}, v_{2n_2})\}$ and $E_2 = \{(v_{2n_2}, v_{1t}), (v_{1t}, v_{2r}), (v_{2r}, v_{1q}), (v_{1q}, v_{2n_2})\}$. Then the bitournament $T' = (T \cup E_1) - E_2$ also has S as a score sequence pair. By setting $T := T'$ and repeating the argument above, we can finally obtain a bitournament T which contains edges (v_{1j}, v_{2n_2}) for all j with $u_j = s_{1j} - 1$. Thus the bitournament $H = T - v_{2n_2}$ has $\{U, S_2 - s_{2n_2}\}$ as a score sequence pair. ■

Based on Proposition 2.2, we can obtain the following iterative algorithm CBT for constructing a bitournament T with $S = \{S_1, S_2\}$ as a score sequence pair. In the algorithm, L is initialized $L = \{j \mid s_{1,j-1} < s_{1j}, j = 2, 3, \cdots, n_1\} \cup \{0, 1, n_1 + 1\}$ and represented by a doubly-linked list and $pre[j] < j < suc[j]$ for each $j \in L$, where $pre[j]$ and $suc[j]$ denote the previous element and the next element of $j \in L$. Note that $U = (u_1, u_2, \cdots, u_{n_1})$ is initialized $U = S_1$ and then maintained to satisfy $0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1}$. L is also maintained to satisfy $L = \{j \mid u_{j-1} < u_j, j = 2, 3, \cdots, n_1\} \cup (0, 1, n_1 + 1\}$. Thus, $u_{pre[j]} = u_{pre[j]+1} = \cdots = u_{j-1} < u_j = u_{j+1} = \cdots = u_{suc[j]-1}$ for each $j \in L$.

Algorithm CBT.

Input: Two sequences of nonnegative integers $S_1 = (s_{11}, s_{12}, \cdots, s_{1n_1})$ with $0 \leqq s_{11} \leqq s_{12} \leqq \cdots \leqq s_{1n_1} \leqq n_2$ and $S_2 = (s_{21}, s_{22}, \cdots, s_{2n_2})$ with $0 \leqq s_{21} \leqq s_{22} \leqq \cdots \leqq s_{2n_2} \leqq n_1$.

Begin
  $L := \{0, 1\}$; **For** $j := 1$ **to** n **do** $u_j := s_{1j}$;
  For $j := 2$ **to** n_1 **do begin**
    If $s_{1,j-1} < s_{1j}$ **then** insert j into L as the last element of L **end**;
  Insert $n_1 + 1$ into L as the last element of L;
  For $h := n_2$ **downto** 1 **do** Add-edge(h)
End.

Procedure Add-edge(h).
Begin
  q := the last element of L;
  while $q - 1 > u_{2h}$ **do** $q := pre[q]$;
  For $j := suc[q]$ **to** n_1 **do begin**
    add edge (v_{1j}, v_{2h}); $u_j := u_j - 1$ **end**;
  $q_{new} := q - u_{2h} + suc[q] - 2$;
  For $j := q_{new} + 1$ **to** $suc[q] - 1$ **do** add edge (v_{2h}, v_{1j});
  For $j := q$ **to** q_{new} **do begin**
    add edge (v_{1j}, v_{2h}); $u_j := u_j - 1$ **end**;
  For $j := 1$ **to** $q - 1$ **do** add edge (v_{2h}, v_{1j});
  If $u_{2h} > q - 1$ **then begin**
    insert $q_{new} + 1$ into L between q and $suc[q]$;
    If $u_{suc[q]} = u_{suc[q]-1}$ **then**
      delete $suc[q]$ from L **end**;
  If $u_q = u_{q-1}$ **then** delete q from L
End.

It is easy to see that Algorithm CBT correctly constructs a bitournament T with $S = \{S_1, S_2\}$ as a score sequence pair of a bitournament and it takes $O(m)$ time ($m=n_1 \cdot n_2$), if we observe that j and suc[j]-1 play roles of x and y in Proposition 2.2 respectively. Thus we have the following theorem.

Theorem 1. For two nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, we can determine in $O(n)$ time whether $S=\{S_1, S_2\}$ is a score sequence pair of a bitournament or not, where $n=n_1+n_2$. Furthermore, if S is so, then a bitournament T with S as a score sequence pair can be constructed in $O(m)$ time, where $m=n_1 \cdot n_2$.

3. Score Sequence Pair Problem of a Bipartite k-Tournament

Let T be a directed graph and V be a vertex set of T. Then T is a *bipartite k-tournament* (*k-bitournament*, for short) if and only if T satisfies the following conditions (C1) through (C3):

(C1) T has two independent vertex sets V_1 and V_2 with $V=V_1 \cup V_2$,

(C2) For some positive integer k and some nonnegative integer k′ with $k' \leqq k$, T has k′ edges (u, v) and $k-k'$ edges (v, u) for any vertex pair $u \in V_1$ and $v \in V_2$,

(C3) T has no edge (u, v) for any vertex pair $u, v \in V_j (j=1, 2)$.

For a pair of nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, $S=\{S_1, S_2\}$ is a *score sequence pair of a k-bitournament* if and only if there is a k-bitournament with two independent vertex sets $V_1=\{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2=\{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ such that $deg^+(v_{1j})=s_{1j}$, $deg^-(v_{1j})=k \cdot n_2-s_{1j}$, $deg^+(v_{2q})=s_{2q}$ and $deg^-(v_{2q})=k \cdot n_1-s_{2q}$ for each $j=1, 2, \cdots, n_1$, $q=1, 2, \cdots, n_2$. The *score sequence pair problem of a k-bitournament* is: Given a pair of nonnegative integer sequences, determine whether it is a score sequence pair of a k-bitournament or not. We can assume without loss of generality that $0 \leqq s_{11} \leqq s_{12} \leqq \cdots \leqq s_{1n_1} \leqq k \cdot n_2$ and $0 \leqq s_{21} \leqq s_{22} \leqq \cdots \leqq s_{2n_2} \leqq k \cdot n_1$.

In this section, we consider the score sequence pair problem of a k-bitournament and give efficient algorithms. We first present an algorithm to determine whether $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament or not in $O(n)$ time. We can obtain the following proposition.

Proposition 3.1. Let $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$ be two sequences of nonnegative integers, where $0 \leqq s_{11} \leqq s_{12} \leqq \cdots \leqq s_{1n_1} \leqq k \cdot n_2$ and $0 \leqq s_{21} \leqq s_{22} \leqq \cdots \leqq s_{2n_2} \leqq k \cdot n_1$. Then $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament if and only if

$$\sum_{j=1}^{t_1} s_{1j} + \sum_{j=1}^{t_2} s_{2j} \geqq k \cdot t_1 \cdot t_2$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$.

Proof. Let $B=(b_1, b_2, \cdots, b_{n_1})$ be a sequence of nonnegative integers with $b_j=n_2-s_{1j}$ for each $j=1, 2, \cdots, n_1$. Then $k \cdot n_2 \geqq b_1 \geqq b_2 \geqq \cdots \geqq b_{n_1} \geqq 0$ and S is a score sequence pair of a k-bitournament if and only if $S'=\{B, S_2\}$ is bipartite k-graphical (i.e., there is a bipartite multigraph with two independent vertex sets $V_1=\{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2=\{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ such that $0 \leqq |\{(u, v) \mid u \in V_1, v \in V_2\}| \leqq k$ ((u, v) is an undirected edge) and such that $deg(v_{1j})=b_j$ and $deg(v_{2q})=s_{2q}$ for each $j=1, 2, \cdots, n_1, q=1, 2, \cdots, n_2$). By the max-flow min-cut theorem by Ford and Fulkerson[5], we can easily prove that S′ is bipartite k-graphical if and only if

$$\sum_{j=1}^{t_1} b_j \leqq \sum_{j=1}^{t_2} s_{2j} + k \cdot t_1(n_2-t_2)$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$. Furthermore,

$$\sum_{j=1}^{t_1} b_j \leqq \sum_{j=1}^{t_2} s_{2j} + k \cdot t_1(n_2-t_2)$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$, if and only if

$$\sum_{j=1}^{t_1} s_{1j} + \sum_{j=1}^{t_2} s_{2j} \geqq k \cdot t_1 \cdot t_2$$

for each $t_1=1, 2, \cdots, n_1$, $t_2=1, 2, \cdots, n_2$, with equality holding for $t_1=n_1$ and $t_2=n_2$.

By the argument above, the proof of this proposition is completed. ■

Similarly Proposition 3.1 can be rewritten as follows: $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament if and only if

$$\sum_{j=1}^{t_1} s_{1j}+\sum_{j=1}^{p(t_1)} s_{2j} \geqq k\cdot t_1\cdot p(t_1)$$

for each $t_1=1, 2, \cdots, n_1$, where $p(t_1)=\max\{j \mid s_{2j}<k\cdot t_1\}$ for each $t_1=1, 2, \cdots, n_1$. We can compute all $p(t_1)$ in $O(n)$ time as follows ($n=n_1+n_2$).

Algorithm Comp-p(t_1).

Step1: $s_{20}:=0$, $s_{2,n_2+1}:=k\cdot n_1$ and $q:=1$.

Step2: **For** $j:=0$ **to** n_2+1 **do** the following (a) and (b).

(a) **while** $s_{2j}\geqq k\cdot q$ **do** $q:=q+1$ and $p(q):=p(q-1)$.

(b) $p(q):=j$.

Thus we can determine whether $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament or not in $O(n)$ time($n=n_1+n_2$).

Next we present an algorithm for actually constructing a k-bitournament based on the following proposition, which can be proved easily.

Proposition 3.2. Let $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$ be two sequences of nonnegative integers, where $0\leqq s_{11}\leqq s_{12}\leqq\cdots\leqq s_{1n_1}\leqq k\cdot n_2$ and $0\leqq s_{21}\leqq s_{22}\leqq\cdots\leqq s_{2n_2}\leqq k\cdot n_1$. Let $U=(u_1, u_2, \cdots, u_{n_1})$ be defined by the following algorithm. Then $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament if and only if $\{U, S_2-s_{2n_2}\}$ is a score sequence pair of a k-bitournament, where $S_2-s_{2n_2}=(s_{21}, s_{22}, \cdots, s_{2,n_2-1})$. Furthermore, $0\leqq u_1\leqq u_2\leqq\cdots\leqq u_{n_1}\leqq k(n_2-1)$.

Algorithm BkT.

Step1: $s_{10}:=0$, $p(0):=0$,

$y(j):=\min\{p(j), n_2-1\}$ for each $j=1, 2, \cdots, n_1$,

$q:=\max\{t \mid \sum_{j=1}^{t} s_{1j}+\sum_{j=1}^{y(t)} s_{2j}=k\cdot t\cdot y(t), 0\leqq t\leqq n_1\}$, $Q_0:=0$ and

$Q_t:=\sum_{j=1}^{t} s_{2j}$ for each $t=1, 2, \cdots, n_2$.

Step2: **For** $j:=1$ **to** q **do** $u_j:=s_{1j}$.

Step3: $R_q:=k\cdot n_1-s_{2n_2}$, $b_q:=k\cdot q\cdot y(q)$ and, **if** $R_q=0$ **then** $x:=q$.

Step4: **For** $j:=q+1$ **to** n_1 **do** the following (a) through (c).

(a) $b_{j-1}:=b_{j-1}+Q_{y(j)}-Q_{y(j-1)}$,
$f_j:=\max\{0, k\cdot j\cdot y(j)-b_{j-1}\}$ and
$h_j:=\min\{s_{1j}-f_j, k, R_{j-1}\}$.

(b) $u_j:=s_{1j}-h_j$, $R_j:=R_{j-1}-h_j$ and $b_j:=b_{j-1}+u_j$.

(c) **If** $R_{j-1}>0$ **and** $R_j=0$ **then** $x:=j$.

Proof. The sufficiency is almost trivial. If $\{U, S_2-s_{2n_2}\}$ is a score sequence pair of a k-bitournament and H is a k-bitournament with $\{U, S_2-s_{2n_2}\}$ as the score sequence pair, then a k-bitournament T obtained from H by adding $s_{1j}-u_j$ edges (v_{1j}, v_{2n_2}) and $k-s_{1j}+u_j$ edges (v_{2n_2}, v_{1j}) for each $j=1, 2, \cdots, n_1$, has $S=\{S_1, S_2\}$ as a score sequence pair.

The necessity can be obtained as follows. It is easy to prove that

$$\sum_{j=1}^{t_1} u_j+\sum_{j=1}^{p(t_1)} s_{2j}\geqq k\cdot t_1\cdot p(t_1)$$

for each $t_1=1, 2, \cdots, n_1$, $q\leqq x\leqq n_1$ and $u_{n_1}\leqq k\cdot(n_2-1)$. Then

$$\sum_{j=1}^{t_1} u_j+\sum_{j=1}^{p(t_1)} s_{2j}\geqq k\cdot t_1\cdot p(t_1)$$

for each $t_1=1, 2, \cdots, n_1$, if and only if

$$\sum_{j=1}^{t_1} u_j+\sum_{j=1}^{t_2} s_{2j}\geqq k\cdot t_1\cdot t_2$$

for each $t_1=1, 2, \cdots, n_1$ and $t_2=1, 2, \cdots, n_2-1$. Furthermore,

$$\sum_{j=1}^{n_1} u_j+\sum_{j=1}^{n_2-1} s_{2j}=k\cdot n_1\cdot(n_2-1).$$

We can prove $0\leqq u_1\leqq u_2\leqq\cdots\leqq u_{n_1}$ as follows. Since $u_j=s_{1j}$ for each $j=1, 2, \cdots, q, x+1, x+2, \cdots, n_1$, we have $0\leqq u_1\leqq u_2\leqq\cdots\leqq u_q$ and $u_{x+1}\leqq u_{x+2}\leqq\cdots\leqq u_{n_1}$. In the following, we prove $u_{t-1}\leqq u_t$ for any $t, q+1\leqq t\leqq x+1$. We consider three cases.

Case 1: Suppose $h_{t-1}=s_{1,t-1}-f_{t-1}$. Then $u_{t-1}\leqq k\cdot y(t-1)$ since $u_{t-1}=h_{t-1}$ and

$$u_{t-1}=k\cdot(t-1)\cdot y(t-1)-\left(\sum_{j=1}^{t-2} u_j+\sum_{j=1}^{y(t-1)} s_{2j}\right).$$

On the other hand,

$$u_t\geqq k\cdot t\cdot y(t)-\left(\sum_{j=1}^{t-1} u_j+\sum_{j=1}^{y(t)} s_{2j}\right).$$

and thus

$$u_t\geqq k\cdot t\cdot y(t-1)-\left(\sum_{j=1}^{t-1} u_j+\sum_{j=1}^{y(t-1)} s_{2j}\right)$$

since

$$\sum_{j=1}^{t_1} u_j + \sum_{j=1}^{p(t_1)} s_{2j} \geqq k \cdot t_1 \cdot p(t_1)$$

for each $t_1 = 1, 2, \cdots, n_1$, if and only if

$$\sum_{j=1}^{t_1} u_j + \sum_{j=1}^{t_2} s_{2j} \geqq k \cdot t_1 \cdot t_2.$$

Thus $u_t - u_{t-1} \geqq k \cdot y(t-1) - u_{t-1} \geqq 0$. Hence we have $u_{t-1} \geqq u_t$.

Case 2: Suppose that $h_{t-1} = k$. Then $u_{t-1} = s_{1,t-1} - k \leqq s_{1t} - k \leqq s_{1t} - h_t = u_t$ since $u_{t-1} = s_{1,t-1} - k$ and $s_{1,t-1} \leqq s_{1t}$. Hence we have $u_{t-1} \leqq u_t$.

Case 3: Suppose that $h_{t-1} = R_{t-1}$. Then $u_t = s_{1t}$ since $x = t-1$. Furthermore, $u_{t-1} = s_{1,t-1} - R_{t-1} < s_{1,t-1} \leqq s_{1t} = u_t$ since $s_{1,t-1} \leqq s_{1t}$. Hence we have $u_{t-1} < u_t$.

By the argument above, $u_q \leqq u_{q+1} \leqq \cdots \leqq u_x \leqq u_{x+1}$, and thus $0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1} \leqq k(n_2 - 1)$. Hence $\{U, S_2 - s_{2n_2}\}$ is a score sequence pair of a k-bitournament by Proposition 3.1. ■

Based on Proposition 3.2, we can obtain the following iterative algorithm CBkT for constructing a k-bitournament T with $S = \{S_1, S_2\}$ as a score sequence pair of a k-bitournament.

Let K_{n_1,n_2} be the complete bipartite directed graph with two independent vertex sets $V_1 = \{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ and $V_2 = \{v_{21}, v_{22}, \cdots, v_{2n_2}\}$ and let $N = (K_{n_1,n_2}, cap)$ be the weighted bipartite graph defined by the capacity function $cap(e) = cap(e') = k$ for $e = (v_{1j}, v_{2q}), e' = (v_{2q}, v_{1j}), j = 1, 2, \cdots, n_1, q = 1, 2, \cdots, n_2$. Since S is a score sequence pair of a k-bitournament, N has a weight of value

$$m = \sum_{j=1}^{n_1} s_{1j} + \sum_{j=1}^{n_2} s_{2j}.$$

For a weight w of N of value m, we create w(e) copies of an edge $e = (v_{1j}, v_{2q})$ and $w(e')$ copies of an edge $e' = (v_{2q}, v_{1j}), j = 1, 2, \cdots, n_1, q = 1, 2, \cdots, n_2$. Then the complete bipartite directed graph H_w obtained in this way has S as a score sequence pair of a k-bitournament. To obtain a k-bitournament, we obtain a maximum weight w of N.

Let h be any integer in $\{1, 2, \cdots, n_2\}$ and $U(h) = \{u_1, u_2, \cdots, u_{n_1}\}$ be a sequence of nonnegative integers with

$$\sum_{j=1}^{n_1} u_j + \sum_{j=1}^{h} s_{2j} = k \cdot n_1 \cdot h \quad \text{and} \quad 0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1} \leqq k \cdot h.$$

In the algorithm, L is initialized $L = \{0\} \cup \{t \mid \sum_{j=1}^{t} s_{1j} + \sum_{j=1}^{y(t)} s_{2j} = k \cdot t \cdot y(t), j = 1, 2, \cdots, n_1\}$ and represented by a linked list and $pre[j] > j > suc[j]$ for each $j \in L$, where pre[j] and suc[j] denote the previous element and the next element of $j \in L$. Note that h and U(h) are initialized $h = n_2$ and $U(h) = S_1$. L is also maintained to satisfy $L = \{0\} \cup (t \mid \sum_{j=1}^{t} u_j + \sum_{j=1}^{y(t)} s_{2j} = k \cdot t \cdot y(t)\}$.

We assume $w(e) = 0$ for $e \in E(K_{n_1,n_2})$ at the beginning, where $E(K_{n_1,n_2})$ is an edge set of K_{n_1,n_2}.

Algorithm CBkT.

Begin
  $L := \{0\}$; $a := 0$; $U(n_2) := S_1$; $Q_0 := 0$;
  For $j := 1$ **to** n_2 **do** $Q_j := Q_{j-1} + S_{2j}$;
  For $j := 1$ **to** n_1 **do begin**
    $y(j) := \min\{p(j), n_2 - 1\}$;
    $a := a + u_j + Q_{y(j)} - Q_{y(j-1)}$;
    If $a = k \cdot j \cdot y(j)$ **then** insert j into L as the first element of L **end**;
  For $h := n_2$ **downto** 1 **do** Add-edge(h)
End.

Procedure Add-edge(h).

Begin
  q := the first element of L;
  while $y(q) = h$ **do begin**
    q := suc[q]; delete pre[q] from L **end**;
  For $j := 1$ **to** q **do begin**
    $w((v_{1j}, v_{2h})) := 0$; $w((v_{2h}, v_{1j})) := k$ **end**;
  $R := k \cdot n_1 - s_{2h}$; $b := k \cdot q \cdot y(q)$;
  For $j := q+1$ **to** n_1 **do begin**
    $b := b + Q_{y(j)} - Q_{y(j-1)}$;
    $f := \max\{0, k \cdot j \cdot y(j) - b\}$;
    $t := \min\{u_j - f, k, R\}$; $u_j := u_j - t$;
    $w((v_{1j}, v_{2h})) := t$;
    $w((v_{2h}, v_{1j})) := k - t$; $R := R - t$; $b := b + u_j$
    end;
  For $j := 1$ **to** n_1 **do** $y(j) := \min\{y(j), h-1\}$
End.

It is easy to see that Algorithm CBkT correctly constructs a k-bitournament T with $S = \{S_1, S_2\}$ as a score sequence pair and that it takes $O(n_1 \cdot n_2)$ time. Thus we have the following theorem.

Theorem 2. For two nonnegative integer sequences $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$, we can determine in O(n) time whether $S=\{S_1, S_2\}$ is a score sequence pair of a k-bitournament or not, where $n=n_1+n_2$. Furthermore, if S is so, then a k-bitournament with S as a score sequence pair can be constructed in O(m) time, where $m=n_1 \cdot n_2$.

4. Concluding Remarks

We have considered the score sequence prob lem of a bipartite tournament and of a bipartite k-tournament, and given optimal algorithms. Furthermore, we have found that there is an intimate relation between the bipartite graphical degree sequence problem[12] and the score sequence pair problem of a bipartite tournament. Based on the relation, we have found the characterization of score sequence of a bipartite k-tournament.

In the following, we present another characterization of a score sequence of a bipartite tournament immediately based on Proposition 3.2.

Proposition 4.1. Let $S_1=(s_{11}, s_{12}, \cdots, s_{1n_1})$ and $S_2=(s_{21}, s_{22}, \cdots, s_{2n_2})$ be two sequences of nonnegative integers, where $0 \leqq s_{11} \leqq s_{12} \leqq \cdots \leqq s_{1n_1} \leqq n_2$ and $0 \leqq s_{21} \leqq s_{22} \leqq \cdots \leqq s_{2n_2} \leqq n_1$. Let $U=(u_1, u_2, \cdots, u_{n_1})$ be defined by Algorithm BkT with k=1. Then $S=\{S_1, S_2\}$ is a score sequence pair of a bipartite tournament if and only if $\{U, S_2 - s_{2n_2}\}$ is a score sequence pair of a bipartite tournament, where $S_2 - s_{2n_2}=(s_{21}, s_{22}, \cdots, s_{2,n_2-1})$. Furthermore, $0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_{n_1} \leqq k(n_2-1)$.

A bipartite tournament with $S=\{S_1, S_2\}$ as a score sequence pair can also be constructed in $O(n_1 \cdot n_2)$ time based on Proposition 4.1.

Finally, we will consider other variations of the score sequence problem of tournaments as our further investigations.

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