Bipartite Digraphical Degree Sequence Problem

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Masaya TAKAHASHI (Department of Office Information Systems, Fukuoka Jr. College of Technology.)

SUMMARY

A sequence of nonnegative integers $S=(s_1, s_2, \cdots, s_n)$ is *graphical* if there is a graph with vertices $v_1, v_2, \cdots, v_n$ such that $\deg(v_j)=s_j$ for each $j=1, 2, \cdots, n$. The graphical degree sequence problem is: Given a sequence of nonnegative integers, determine whether it is graphical or not. In this paper, we consider the k-partite multidigraphical degree sequence problem and bipartite digraphical one, and give efficient algorithms for them.

Key words : *k-partite directed multigraph, bipartite directed graph, nonnegative integers and efficient algorithms*

1. Introduction

A sequence of nonnegative integers $S=(s_1, s_2, \cdots, s_n)$ is *graphical* if there is a graph with vertices $v_1, v_2, \cdots, v_n$ such that $\deg(v_j)=s_j$ for each $j=1, 2, \cdots, n$ ($\deg(v_j)$ is the degree of v_j). The graphical degree sequence problem is: Given a sequence of nonnegative integers, determine whether it is graphical or not. The graphical degree sequence problem was first considered by Havel[7] and then considered by Erdös and Gallai[4] and Hakimi[5]. (These can be also found in standard books of graph theory[1, 6].)

Many variations can be considered. For example, if we admit multigraphs, then the multigraphical version is obtained. We recently studied variations described below[8, 9]. A set of sequences of nonnegative integers $\{S_1, S_2, \cdots, S_k\}$ with $S_j=(s_{j1}, s_{j2}, \cdots, s_{jn_j})$ is *k-partite graphical* (*k-partite multigraphical*) if there is a k-partite graph (k-partite multigraph) of k independent vertex sets $\{V_1, V_2, \cdots, V_k\}$ with $V_j=\{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ such that $\deg(v_{jq})=s_{jq}$ for each $j=1, 2, \cdots, k$, $q=1, 2, \cdots, n_j$. The k-partite graphical (multigraphical) degree sequence problem is defined as follows: Given a set of sequences of nonnegative integers, determine whether it is k-partite graphical (multigraphical) or not. Takahashi, Imai and Asano considered the graphical, multigraphical, bipartite graphical, bipartite multigraphical and k-partite multigraphical degree sequence problem and gave efficient algorithms for them[9].

The problems stated above are all about undirected graphs. Directed versions can also be obtained[2]. For example, a pair of nonnegative integer sequences $\{S^+, S^-\}$ is *digraphical* (*multidigraphical*) if there is a directed graph (directed multigraph) with vertices $v_1, v_2, \cdots, v_n$ such that $\deg^+(v_j)=s_j^+$ and $\deg^-(v_j)=s_j^-$ for each $j=1, 2, \cdots, n$ ($\deg^+(v_j)$ and $\deg^-(v_j)$ are the outdegree and indegree of v_j respectively), where $S^+=(s_1^+, s_2^+, \cdots, s_n^+)$ and $S^-=(s_1^-, s_2^-, \cdots, s_n^-)$. The digraphical (multidigraphical) degree sequence problem is defined similarly: Given a pair of nonnegative integer sequences, determine whether it is digraphical (multidigraphical) or not. We also considered them and presented efficient algorithms[9].

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In this paper, we consider *k-partite multidigraphical* degree sequence problem and *bipartite digraphical* one, and give efficient algorithms.

2. k-partite Multigraphical Degree Sequence Problem

In this section, we consider the k-partite multidigraphical (kpm-digraphical, for short) degree sequence problem: Given a set of k pairs of nonnegative integer sequences $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ for each $j=1, 2, \cdots, k$, determine whether it is kpm-digraphical (that is, there is a k-partite directed multigraph (kpm-digraph, for short) of k independent vertex sets $\{V_1, V_2, \cdots, V_k\}$ with $V_j=\{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ such that $\deg^+(v_{jq})=s_{jq}^+$ and $\deg^-(v_{jq})=s_{jq}^-$ for each $j=1, 2, \cdots, k, q=1, 2, \cdots, n_j$), where $k \geqq 2$ and $n=n_1+n_2+\cdots+n_k$.

Let $x_j^+=\sum_{q=1}^{n_j} s_{jq}^+$ and $x_j^-=\sum_{q=1}^{n_j} s_{jq}^-$ for each $j=1, 2, \cdots, k$, $x^+=x_1^++x_2^++\cdots+x_k^+$ and $x^-=x_1^-+x_2^-+\cdots+x_k^-$. Then we assume $x^+=x^-$, because otherwise $\{(S_j^+, S_j^-) | j=1, 2, \cdots, k\}$ is not kpm-digraphical.

We first consider the k=2 case (i.e., bipartite multidigraphical degree sequence problem). The problem can be solved easily. (The problem can be solved in a similar way as the bipartite multigraphical degree sequence problem[9].)

Proposition 2.1. Let $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ for each $j=1, 2$, be a pair of nonnegative integer sequences. Then $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite multidigraphical if and only if $x_1^+=x_2^-$ and $x_1^-=x_2^+$.

Based on Proposition 2.1, the following bipartite directed multigraph construction algorithm is immediately obtained: If $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite multidigraphical, then

(1) add $\min(s_{11}^+, s_{21}^-)$ edges from vertex v_{11} to vertex v_{21} and do recursively for $\{(S_1^{+\prime}, S_1^-), (S_2^+, S_2^{-\prime})\}$, where

$$S_1^{+\prime}=\begin{cases}(s_{11}^+-s_{21}^-, s_{12}^+, \cdots, s_{1n_1}^+) & \text{if } s_{11}^+>s_{21}^- \\ (s_{12}^+, \cdots, s_{1n_1}^+) & \text{otherwise}\end{cases}$$

and

$$S_2^{-\prime}=\begin{cases}(s_{21}^--s_{11}^+, s_{22}^-, \cdots, s_{2n_2}^-) & \text{if } s_{11}^+<s_{21}^- \\ (s_{22}^-, \cdots, s_{2n_2}^-) & \text{otherwise,}\end{cases}$$

(2) add $\min(s_{11}^-, s_{21}^+)$ edges from vertex v_{21} to vertex v_{11} and do recursively for $\{(S_1^+, S_1^{-\prime}), (S_2^{+\prime}, S_2^-)\}$, where

$$S_1^{-\prime}=\begin{cases}(s_{11}^--s_{21}^+, s_{12}^-, \cdots, s_{1n_1}^-) & \text{if } s_{11}^->s_{21}^+ \\ (s_{12}^-, \cdots, s_{1n_1}^-) & \text{otherwise}\end{cases}$$

and

$$S_2^{+\prime}=\begin{cases}(s_{21}^+-s_{11}^-, s_{22}^+, \cdots, s_{2n_2}^+) & \text{if } s_{11}^-<s_{21}^+ \\ (s_{22}^+, \cdots, s_{2n_2}^+) & \text{otherwise.}\end{cases}$$

Theorem 1. For a set of two pairs of nonnegative integer sequences $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ for each $j=1, 2$, we can determine in $O(n)$ time whether $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite multidigraphical and, if so, a bipartite directed multigraph G with $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ as a set of degree sequences can be constructed in $O(n)$ time ($n=n_1+n_2$).

Next we consider the $k \geqq 3$ case. The kpm-digraphical degree sequence problem can also be solved easily. The following proposition can be obtained in a similar way as the multigraphical degree sequence problem[9].

Proposition 2.2. Let $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ be a set of k pairs of nonnegative integer sequences with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$. Then $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ is kpm-digraphical if and only if $x_j^++x_j^- \leqq X^+(=X^-)$.

Based on Proposition 2.2, we can determine whether $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ is kpm-digraphical or not in $O(n)$ time, where $n=n_1+n_2+\cdots+n_k$.

To construct a kpm-digraph G with $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ as a set of degree sequences, we first construct a condensed directed multigraph H with k vertices $w_1, w_2, \cdots, w_k$ which has $X^+=(x_1^+, x_2^+, \cdots, x_k^+)$ and $X^-=(x_1^-, x_2^-, \cdots, x_k^-)$ as a pair of degree sequences. H can be obtained in $O(k)$ time based on the directed multigraph construction algorithm[9]. Suppose that there are just

h_{jq} edges from w_j to w_q and h_{qj} edge from w_q to w_j in H. Restricting the vertex set of G to V_j and V_q, we construct a bipartite directed multigraph G_{jq} with h_{jq} edges from V_j to V_q and h_{qj} edges from V_q to V_j based on the bipartite directed multigraph construction algorithm described above and then modify the pairs of degree sequences of (S_j^+, S_j^-) and (S_q^+, S_q^-). Repeating this procedure iteratively, we can construct a kpm-digraph G in O(n) time.

Theorem 2. For a set of k pairs of nonnegative integer sequences $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$, we can determine in O(n) time whether $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ is kpm-digraphical $(n=n_1+n_2+\cdots+n_k)$ and, if so, a kpm-digraph G with $\{(S_j^+, S_j^-) \mid j=1, 2, \cdots, k\}$ as a set of degree sequences can be constructed in O(n) time.

3. Bipartite Digraphical Degree Sequence Problem

In this section, we consider the bipartite digraphical degree sequence problem: Given a set of two pairs of nonnegative integer sequences $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ for each $j=1, 2$, determine whether it is bipartite digraphical (that is, there is a bipartite directed graph of two independent vertex sets $\{V_1, V_2\}$ with $V_j=\{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ such that $deg^+(v_{jq})=s_{jq}^+$ and $deg^-(v_{jq})=s_{jq}^-$ for each $j=1, 2, q=1, 2, \cdots, n_j$), where $n=n_1+n_2$.

Let $x_j^+=\sum_{q=1}^{n_j} s_{jq}^+$ and $x_j^-=\sum_{q=1}^{n_j} s_{jq}^-$ for each $j=1, 2$, $x^+=x_1^++x_2^+$ and $x^-=x_1^-+x_2^-$. Then we assume $x^+=x^-$, $x_j^++x_j^- \leqq x^+(=x^-)$ for each $j=1, 2$, because otherwise $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is not bipartite digraphical.

This problem can be solved easily. (This problem can be solved in a similar way as the bipartite graphical degree sequence problem[9].)

We assume the following conditions (C1) through (C3) without loss of generality:

(C1) $n_2 \geqq s_{11}^+ \geqq s_{12}^+ \geqq \cdots \geqq s_{1n_1}^+$ and $n_1 \geqq s_{21}^+ \geqq s_{22}^+ \geqq \cdots \geqq s_{2n_2}^+$,

(C2) $s_{1j}^- \leqq n_2$ for each $j=1, 2, \cdots, n_1$, and $s_{2j}^- \leqq n_1$ for each $j=1, 2, \cdots, n_2$,

(C3) $s_{1p_1(1)}^- \leqq s_{1p_1(2)}^- \leqq \cdots \leqq s_{1p_1(n_1)}^- \leqq n_2$, $s_{2p_2(1)}^- \leqq s_{2p_2(2)}^- \leqq \cdots \leqq s_{2p_2(n_2)}^- \leqq n_1$ and for some permutations p_1 on $\{1, 2, \cdots, n_1\}$ and p_2 on $\{1, 2, \cdots, n_2\}$.

Then the following proposition can be obtained in a similar way as the bipartite graphical degree sequence problem[9].

Proposition 3.1. Let $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ be a pair of nonnegative integer sequences for each $j=1, 2$, with $x_1^+=x_2^-$, $x_1^-=x_2^+$ and satisfying the conditions (C1) through (C3) described above. Then $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite digraphical if and only if

$$\sum_{t=1}^{j} s_{1t}^+ \leqq j(n_2-q)+\sum_{t=1}^{q} s_{2p_2(t)}^-$$

for each $j=1, 2, \cdots, n_1$, $q=1, 2, \cdots, n_2$, and

$$\sum_{t=1}^{j} s_{2t}^+ \leqq j(n_1-q)+\sum_{t=1}^{q} s_{1p_1(t)}^-$$

for each $j=1, 2, \cdots, n_2$, $q=1, 2, \cdots, n_1$.

Based on Proposition 3.1, we can determine whether $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite digraphical or not in O(n) time $(n=n_1+n_2)$ by a similar way as the bipartite graphical degree sequence problem[9]. Thus we have the following theorem.

Theorem 3. For a set of two pairs of nonnegative integer sequences $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ with $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ for each $j=1, 2$, we can determine whether $\{(S_1^+, S_1^-), (S_2^+, S_2^-)\}$ is bipartite digraphical or not in O(n) time, where $n=n_1+n_2$.

Furthermore, we present an algorithm for actually constructing a bipartite directed graph for a given bipartite digraphical set of degree sequences based on the following proposition, which can be obtained easily in a similar way as the bipartite graphical degree sequence problem[9].

Proposition 3.2. Let $S_j^+=(s_{j1}^+, s_{j2}^+, \cdots, s_{jn_j}^+)$ and $S_j^-=(s_{j1}^-, s_{j2}^-, \cdots, s_{jn_j}^-)$ be a pair of nonnegative integer sequences for each $j=1, 2$, with $x_1^+=x_2^-$, $x_1^-=x_2^+$, $s_{11}^+ \leqq s_{12}^+ \leqq \cdots \leqq s_{1n_1}^+ \leqq n_2$, $s_{21}^+ \leqq s_{22}^+ \leqq \cdots \leqq s_{2n_2}^+ \leqq n_1$ and satisfying the conditions (C2) and (C3) described above. Let $T_1=(t_{11}, t_{12}, \cdots,$

t_{1n_1}) and $T_2=(t_{21}, t_{22}, \cdots, t_{2n_2})$ be defined by using $z_1=s^-_{2p_2(n_2)}$, $z_2=s^-_{1p_1(n_1)}$, $r_1=\min\{j \mid s^+_{1j}=s^+_{1,n_1-z_1+1}\}$, $r_2=\min\{j \mid s^+_{2j}=s^+_{2,n_2-z_2+1}\}$, $y_1=\max\{j \mid s^+_{1j}=s^+_{1,n_1-z_1+1}\}$ and $y_2=\max\{j \mid s^+_{2j}=s^+_{2,n_2-z_2+1}\}$ as follows.

$$t_{1j}=\begin{cases} s^+_{1j}-1 & \text{if } y_1+1\leqq j\leqq n_1 \text{ or } r_1\leqq j\leqq r_1+y_1-n_1+z_1-1, \\ s^+_{1j} & \text{if } 1\leqq j\leqq r_1-1 \text{ or } r_1+y_1-n_1+z_1\leqq j\leqq y_1, \end{cases}$$

and

$$t_{2j}=\begin{cases} s^+_{2j}-1 & \text{if } y_2+1\leqq j\leqq n_2 \text{ or } r_2\leqq j\leqq r_2+y_2-n_2+z_2-1, \\ s^+_{2j} & \text{if } 1\leqq j\leqq r_2-1 \text{ or } r_2+y_2-n_2+z_2\leqq j\leqq y_2. \end{cases}$$

Then $\{(S^+_1, S^-_1), (S^+_2, S^-_2)\}$ is bipartite digraphical if and only if $\{(T_1, S^-_1-s^-_{1p_1(n_1)}), (T_2, S^-_2-s^-_{2p_2(n_2)})\}$ is bipartite digraphical, where $S^-_1-s^-_{1p_1(n_1)}=(s^-_{1p_1(1)}, s^-_{1p_1(2)}, \cdots, s^-_{1p_1(n_1-1)})$ and $S^-_2-s^-_{2p_2(n_2)}=(s^-_{2p_2(1)}, s^-_{2p_2(2)}, \cdots, s^-_{2p_2(n_2-1)})$. Furthermore, $t_{11}\leqq t_{12}\leqq \cdots \leqq t_{1n_1}$ and $t_{21}\leqq t_{22}\leqq \cdots \leqq t_{2n_2}$.

Based on Proposition 3.2, we can obtain easily an O(m) time algorithm for constructing a bipartite directed graph which is similar to the bipartite graph construction algorithm, where $m=x^+(=x^-)$. Thus we have the following theorem.

Theorem 4. For a bipartite digraphical set of degree sequences $\{(S^+_1, S^-_1), (S^+_2, S^-_2)\}$ with $S^+_j=(s^+_{j1}, s^+_{j2}, \cdots, s^+_{jn_j})$ and $S^-_j=(s^-_{j1}, s^-_{j2}, \cdots, s^-_{jn_j})$ for each $j=1, 2$, a bipartite directed graph G with $V_j=\{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ having $\{(S^+_1, S^-_1), (S^+_2, S^-_2)\}$ as a set of degree sequences can be obtained in O(m) time, where $m=x^+(=x^-)$.

4. Concluding Remarks

We have considered k-partite multidigraphical degree sequence problem and bipartite digraphical one, and given the following results:

(1) For bipartite multidigraphical degree sequence problem, the O(n) time algorithm for determination and construction,

(2) For k-partite multidigraphical one ($k\geqq 3$), the O(n) time algorithm for determination and construction,

(3) For bipartite digraphical one, the O(n) time algorithm for determination and the O(m) time algorithm for construction.

The k-partite graphical degree sequence problem is polynomially solvable based on maximum matching algorithms[3]. We want to consider the more efficient algorithm for solving the k-partite graphical one ($k\geqq 3$) for our further investigation.

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