Some Considerations of Erdos-Sos Conjecture

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Abstract

Erdös and Sòs conjectured in 1963 that if G is a graph having n vertices and m edges satisfying $m > n(k-1)/2$, then G contains every tree having k edges. For example, if G is a complete graph, it is clear that G contains every tree having $n-1$ edges(i.e., the conjecture is true in this case) , since G satisfies $m = n(n-1)/2$ and $k \leqq n-1$, and there is an edge (u,v) for any two vertices $u,v \in G(u \neq v)$. Furthermore, some partial results have been obtained in other special cases. However, this conjecture is still open in general graphs satisfying $m > n(k-1)/2$. In this paper, we prove that the conjecture is true in regular graphs and in complete bipartite graphs.

Keywords: *Erdös-Sòs conjecture*, *regular graph*, *complete bipartite graph*, *tree*

1. Introduction

We shall use standard graph theory definitions and notations throughout this paper. Similarly, we consider only finite and undirected graphs with n vertices, m edges, no loop and no multiple edges.

Let T be any tree and $V[T]$ be the vertex set of T.
For any two distinct vertices $u, v \in V[T]$, we define that the *distance from u to v* (*distance*, for short) is the length of the shortest path from u to v, and denote $d[u,v]$. For some vertex $v \in V[T]$ and any vertex $v' \in V[T]-\{v\}$, $d[v, v']$ is maximum *if and only if* $d[v, v']$ is called the *depth of T with respect to v*.

In 1959, Erdös and Gallai[5] proved that every graph with $m > n(k-1)/2$ contains a path of length k. Based on this result, Erdös and Sòs made the following conjecture in 1963: "If G is a graph with $m > n(k-1)/2$, then G contains every tree T with k edges."

This conjecture is still open. However, some partial results have been obtained. For graphs with large girth, Brandt and Dobson proved that this conjecture is true[3,4]. This conjecture also has been proved for many particular families (caterpillars, paths, stars, double-stars, trees obtained from two stars by joining their centers by a path of length 2 or 3, trees with a vertex adjacent to equal or greater than $(k-1)/2$ leaves, spiders of diameter$\leqq 4$, and others) of trees[2-7,9,11]. For graphs with $k=n-1$, Sauer and Spencer[8] and Zhou[13] proved independently that this conjecture is true. For graphs with $k=n-2$ and with $k=n-3$, Slater[10], Yap[12] and Woźniak[11] proved that this conjecture is true, respectively, Furthermore, for $k \leqq 6$ case, Woźniak[11] proved that this conjecture is true.

In this paper, we prove that this conjecture is true in several specified graphs. If $m=0$ then $k=0$ and trivial. Hence we assume that $m \geqq 1$ and $k \geqq 1$ throughout this paper.

Let $X = (x[1], x[2], \cdots, x[n])$ be a nonnegative integer sequence, c be any integer, h be any integer $1 \leqq h \leqq n$ and g be any integer $1 \leqq g \leqq n$. Then $\Sigma[x[j]+c : j=h,g]$, $\Sigma[x[j]+c : j=h, h-1]$ and $\Sigma[x[j]+c : j=h+1, h]$ mean

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$$\Sigma[x[j]+c:j=h,g]=\sum_{j=h}^{g}(x[j]+c),$$

$$\Sigma[x[j]+c:j=h,h-1]=\sum_{j=h}^{h-1}(x[j]+c)=0$$

and

$$\Sigma[x[j]+c:j=h+1,h]=\sum_{j=h+1}^{h}(x[j]+c)=0,$$

respectively, throughout this paper.

2. Regular Graph

Assume that G is a regular graph with $deg_G[v]=r$ and with n vertices throughout this section, where v is any vertex of G and r is an integer with $1\leqq r<n-1$. Then G has $rn/2$ edges, and we can obviously obtain any tree T with r edges by the following proposition.

Proposition2. 1[1]: Let T be any tree with k edges, and G be a graph with $deg_G[v]\geqq k$. Then T is a subgraph of G, where v is any vertex of G.

By the argument above, we can prove that the conjecture is true in regular graphs, and we can obtain the following theorem.

Theorem1: For a regular graph G with $deg_G[v]=r$ and with n vertices, G contains every tree T with r edges.

3. Complete Bipartite Graph $K(p,q)$

Let $K(p,q)$ be a complete bipartite graph with $V_1=\{v_1[1], v_1[2],\cdots,v_1[p]\}$, $V_2=\{v_2[1],v_2[2],\cdots,v_2[q]\}$, where V_1 and V_2 are distinct vertex sets of $K(p,q)$ with $V_1\cap V_2=\phi$. Let n and m be a number of vertices of $K(p,q)$ and a number of edges of $K(p,q)$, respectively. Then $n=p+q$ and $m=p\cdot q$ hold. If $p=0$ or $q=0$ then $m=0$ and $k=0$ hold. Hence we assume that $p\geqq 1$ and $q\geqq 1$ throughout this section. Then $k\geqq 1$ holds. Furthermore, if $q=1$ then the *girth* of $K(p,q)$ is greater than 5, and therefore, $K(p,q)$ contains any tree with k edges[3,4]. Thus we can assume that $p\geqq 2$ and $q\geqq 2$. Then the *girth* of $K(p,q)$ is 4 clearly.

Suppose that $p\geqq q$ and $p=t\cdot q$. Then t is a real number with $t\geqq 1$, and $m=t\cdot q^2$ holds.

We consider the integer k satisfying

$m=t\cdot q^2>(p+q)\cdot(k-1)/2=q\cdot(t+1)\cdot(k-1)/2.$

Then we can obtain

$k<2\cdot t\cdot q/(t+1)+1.$ ········· (A 1)

Since $1\leqq 2\cdot t/(t+1)<2$ and k is a largest integer satisfying the condition (A 1), we can obtain

$q\leqq k\leqq 2\cdot q$ ········· (A 2)

and

$k\geqq 2\cdot t\cdot q/(t+1).$ ········· (A 3)

Furthermore, we consider the relation of k and $t\cdot q$. If $t>2$ then we can obtain $k<t\cdot q$ by the condition (A2). Thus we can assume that $1\leqq t\leqq 2$. Furthermore, we assume that

$2\cdot t\cdot q/(t+1)=x+\varepsilon,$ ········· (A 4)

where x is the largest integer satisfying $x\leqq 2\cdot t\cdot q/(t+1)$ and ε is a real number satisfying $0\leqq\varepsilon<1$. Then we have $x\geqq 1$. We consider two cases.

Case1: Suppose that $\varepsilon=0$. Then we have $2\cdot t\cdot q/(t+1)=x$ and $k=x$. Thus we have $t\cdot q=(t+1)\cdot x/2$ and $(t+1)x/2\geqq x$ since $1\leqq t\leqq 2$. Hence we can obtain $k\leqq t\cdot q$.

Case2: Suppose that $0<\varepsilon<1$. Then we have $k=x+1$ and $t>1$. Thus we have $t\cdot q=(t+1)\cdot(x+\varepsilon)/2$. Assume that $k\geqq t\cdot q+1$. Then we must have $\varepsilon\leqq-(t-1)\cdot x/(t+1)$, since $t\cdot q+1=(t+1)\cdot(x+\varepsilon)/2+1\leqq k=x+1$. However, we can obtain $\varepsilon<0$, since $1<t\leqq 2$ and $x\geqq 1$, and a contradiction.

By the argument above, we can obtain

$k\leqq t\cdot q.$ ········· (A 5)

Thus, by (A2) and (A5), we can obtain

$q\leqq k\leqq a\cdot q.$ ········· (A 6)

where $a=max\{2,t\}$.

Let T be any tree with k edges, $u[0]$ be a vertex of $V[T]$ such that $deg_T[u[0]]$ is maximum, and d be a depth of T with respect to $u[0]$. Furthermore, let d_1 be the largest integer satisfying $d_1\leqq d/2$. Then, for each $j=1,2,\cdots,d$, let $U[j]$ be a vertex set satisfying the following (1) through (3):

(1)For each $j=1,2,\cdots,d_1$,

$U[j]=\{u\mid u\in V[T],d[u[0],u]=2\cdot j\}$,

(2)For each $j=d_1+1, d_1+2,\cdots,d$,

$U[j]=\{u\mid u\in V[T], d[u[0],u]=2\cdot(j-d_1)-1\}$,

(3)For each $j=1,2,\cdots,d$,

$U[j]=\{u[j,1], u[j,2], \cdots, u[j, |U[j]|]\}$.

Then $V[T]=\{u[0]\}\cup U[1]\cup U[2]\cup\cdots\cup U[d]$ holds and we have

$deg_T[u[0]]+\Sigma[\Sigma[deg_T[u[j,i]]:i=1, |U[j]|];j=1,d]=2\cdot k$ ········· (A 7)

$deg_T[u[0]]+\Sigma[\Sigma[deg_T[u[j,i]]:i=1, |U[j]|];j=1,d_1]=k$, ……… (A 8)

and

$\Sigma[\Sigma[deg_T[u[j,i]]:i=1, |U[j]|];j=d_1+1,d]=k.$ ……… (A 9)

Let $b_1=\Sigma[|U[j]|:j=d_1+1,d]$ and $b_2=\Sigma[|U[j]|:j=1,d_1]$. Since $k\geqq 1$, we have $d\geqq 1$ and $|U[1]|\geqq 1$, and therefore, we have $b_1\geqq 1$. Furthermore, since $b_1+b_2=k$, we have

$b_1=k-b_2=a\cdot q-b_2\leqq t\cdot q.$ ……… (A 10)

Similarly, we have $b_2=k-b_1=a\cdot q-b_1\leqq a\cdot q-1\leqq t\cdot q-1$, and therefore,

$b_2+1=k-b_1+1=a\cdot q-b_1+1\leqq a\cdot q\leqq t\cdot q.$ ……… (A11)

Then we consider the following two cases.

Casel: Suppose that $b_1>q$. Let d_2 be the largest integer satisfying $d_2\leqq(d-1)/2$. Furthermore, let $U[0]=\{u[0]\}$ and $B[j]=\{(u,v)\mid u\in U[j], v\in U[d_1+j+1], (u,v)\in T\}$ for each $j=0,1,2,\cdots,d_2$. Then, by the property of a tree, we have

$b_1=\Sigma[|U[j]|:j=d_1+1,d]$

$=\Sigma[|B[j]|:j=d_1+1,d]\leqq k.$

Thus, by(A9) and (A10), we have

$\Sigma[\Sigma[(deg_T[u[j,i]]-1):i=1, |U[j]|];j=d_1+1,d]$

$=k-b_1=b_2.$ ……… (A 12)

Let $B'[j]=\{(u,v)\mid u\in U[d_1+j], v\in U[j],(u,v)\in T\}$ for each $j=1,2,\cdots,d_1$. Then, by the property of a tree, we have

$b_2=\Sigma[|U[j]|:j=1,d_1]=\Sigma[|B'[j]|:j=1,d_1]$.

Thus, by (A8) and (A11), we have

$deg_T[u[0]]+\Sigma[\Sigma[(deg_T[u[j,i]]-1):i=1, |U[j]|];j=1,d_1]$

$=k-(b_2+1)=b_1.$ ……… (A 13)

Let $z_1[1]=0$, let $z_1[j]=\Sigma[|U[d_1+i]|:i=1,j-1]$ for each $j=d_1+1, d_1+2,\cdots, d$, and let $z_2[j]=\Sigma[|U[i]|:i=0,j-1]$ for each $j=1,2,\cdots,d_1$. Then we can separate V_1 and V_2 as the following (a) through(c):

(a) Set $W_1[j]=\{v_1[z_1[j]+i]\mid i=1,2,\cdots, |U[d_1+j]|\}$ for each $j=1,2,\cdots,d-d_1$,

(b) Set $W_2[j]=\{v_2[z_2[j]+i]\mid i=1,2,\cdots, |U[j]|\}$ for each $j=1,2,\cdots,d_1$,

(c) Set $W_2[0]=\{v_2[1]\}$.

Clealy, there is the one-to-one correspondence between a vertex set $W_1[j]$ and the set $U[d_1+j]$ for each $j=1,2,\cdots,d-d_1$, and between a vertex set $W_2[j]$ and $U[j]$ for each $j=0,1,2,\cdots,d_1$. That is, there is the one-to-one correspondence between the vertex $v_1[z_1[j]+i]$ and the vertex $u[d_1+j,i]$ for each $i=1,2,\cdots, |U[d_1+j]|$, $j=1,2,\cdots,d-d_1$, and between a vertex $v_2[z_2[j]+i]$ and the vertex $u[j,i]$ for each $i=1,2,\cdots, |U[j]|$, $j=1,2,\cdots,d_1$. Then, since $K(p,q)$ is a complete bipartite graph, $K(p,q)$ has edge sets corresponding to $B[j]$ and $B'[i]$ for each $j=0,1,2,\cdots,d_2$, $i=1,2,\cdots,d_1$. Hence $K(p,q)$ contains a tree T with k edges.

Case2: Suppose that $b_1\leqq q$. Then we can also prove that $K(p,q)$ contains a tree T with k edges by the similar argument of Casel as follows. Let $z_2[1]=0$, let $z_2[j]=\Sigma[|U[d_1+i]|:i=1,j-1]$ for each $j=d_1+1, d_1+2,\cdots,d$, and let $z_1[j]=\Sigma[|U[i]|:i=0,j-1]$ for each $j=1,2,\cdots,d_1$. Then we can separate V_1 and V_2 as the following (a') through (c'):

(a') Set $W_2[0]=\{v_1[1]\}$,

(b') Set $W_2[j]=\{v_1[z_1[j]+i]\mid i=1,2,\cdots, |U[j]|\}$ for each $j=1,2,\cdots,d_1$,

(c') Set $W_1[j]=\{v_2[z_2[j]+i]\mid i=1,2,\cdots, |U[d_1+j]|\}$ for each $j=1,2,\cdots,d-d_1$.

Similarly, $K(p,q)$ contains a tree T with k edges.

By the argument above, we can prove that the conjecture is true in complete bipartite graphs, and we can obtain the following theorem.

Theorem2: For a complete bipartite graph $K(p,q)$, $K(p,q)$ contains every tree T with k edges, where k is an integer satisfying $p\cdot q>(p+q)\cdot(k-1)/2$. Furthermore, we can find a tree T with k edges in $O(k)$ time.

4. Examples

In the following, we will show examples for two cases described above.

4.1 Regular Graph

Let G be a regular graph shown as Fig.1.1.Clearly, G has 8 vertices and $deg_G[v]=5$, where v is any vertex of G. Then there are 6 trees with 5 edges shown as Fig.1.2.through 1.7.

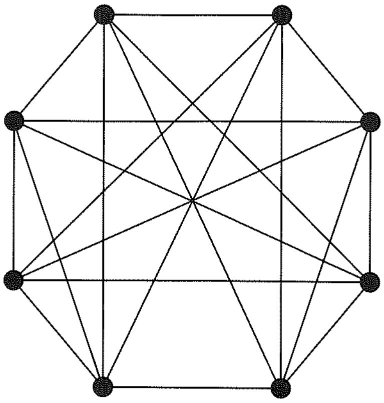

Fig. 1. 1 A regular graph G

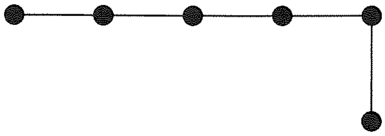

Fig. 1. 2 Type1 tree with 5 edges.

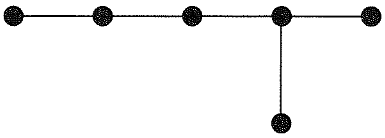

Fig. 1. 3 Type2 tree with 5 edges.

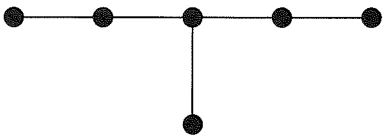

Fig. 1. 4 Type3 tree with 5 edges.

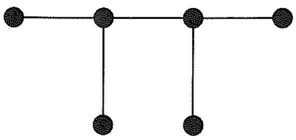

Fig. 1. 5 Type4 tree with 5 edges.

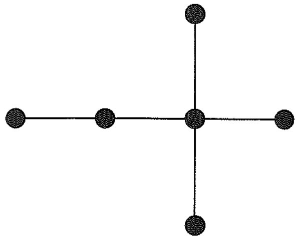

Fig. 1. 6 Type5 tree with 5 edges.

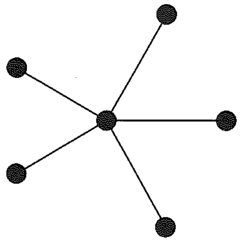

Fig. 1. 7 Type6 tree with 5 edges.

Then G has every tree with 5 edges. See Fig.1.8 through 1.13 as follows. ( : edge of the tree.)

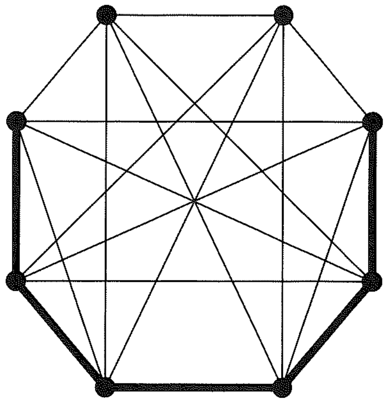

Fig. 1. 8 G has Type1 tree.

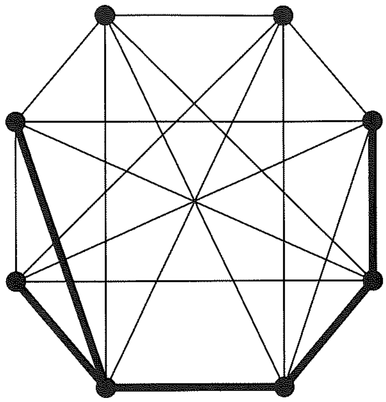

Fig. 1. 9 G has Type2 tree.

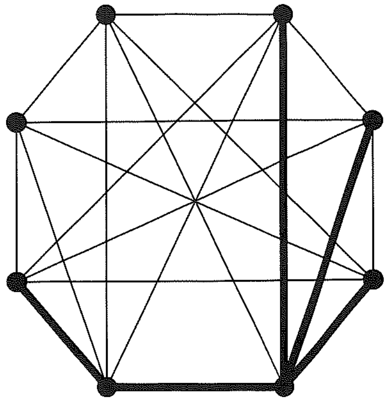

Fig. 1. 12 G has Type5 tree.

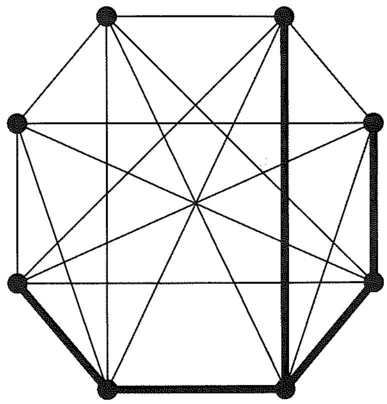

Fig. 1. 10 G has Type3 tree.

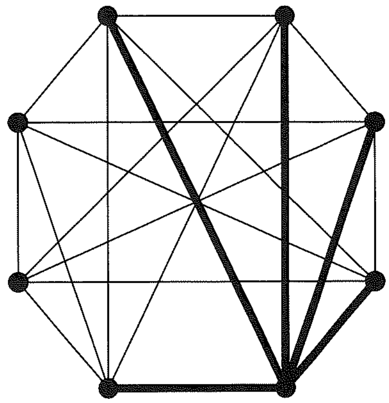

Fig. 1. 13 G has Type6 tree.

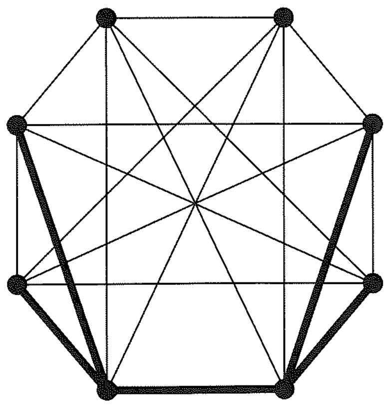

Fig. 1. 11 G has Type4 tree.

4. 2 Complete Bipartite Graph $K(p,q)$

Let $K(6,4)$ be a bipartite graph shown as Fig.2.1. Clearly, we have $n=p+q=10$ and $m=p\cdot q=24$, since $p=6$ and $q=4$. Thus we obtain $k=5$, since $m=24>10\cdot(5-1)/2=20$ and $m=24<10\cdot(6-1)/2=25$. Then there are 6 trees with 5 edges shown as Fig.1.2 through 1.7.

Then $K(6,4)$ has every tree with 5 edges. See Fig.2.2 through 2.7 as follows. (:edge of the tree.)

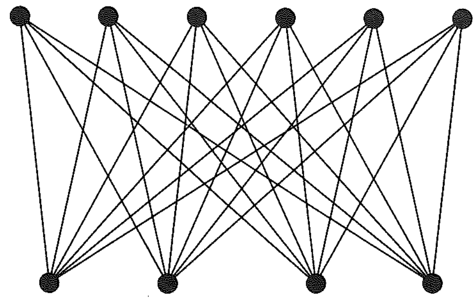

Fig. 2. 1 A bipartite graph $K(6,4)$.

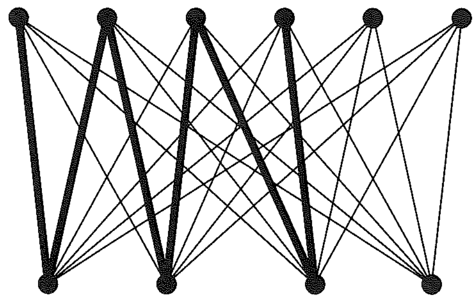

Fig. 2. 2 $K(6,4)$ has Type1 tree.

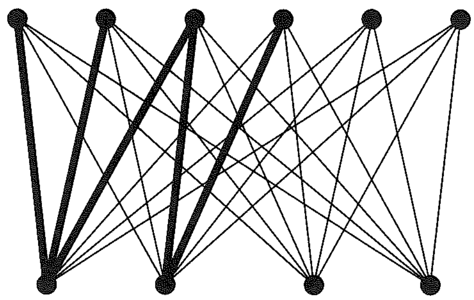

Fig. 2. 3 $K(6,4)$ has Type2 tree.

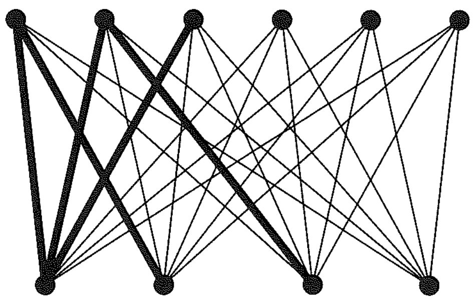

Fig. 2. 4 $K(6,4)$ has Type3 tree.

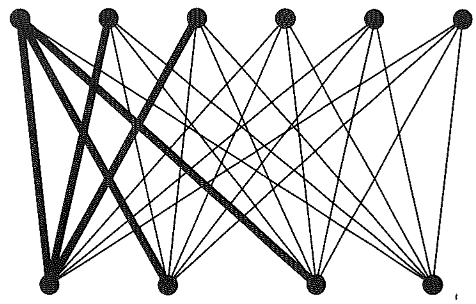

Fig. 2. 5 $K(6,4)$ has Type4 tree.

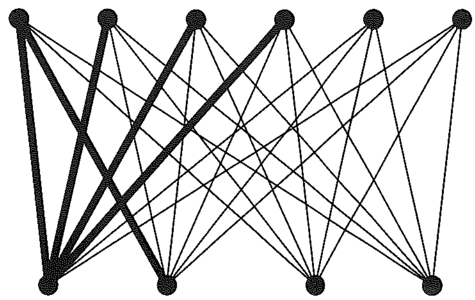

Fig. 2. 6 $K(6,4)$ has Type5 tree.

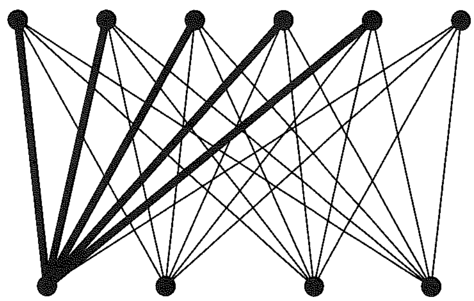

Fig. 2. 7 $K(6,4)$ has Type6 tree.

5. Concluding Remarks

We have shown that Erdös-Sös conjecture is true in regular graphs and complete bipartite graphs. Furthermore, we have shown an optimal algorithm for finding a tree satisfying the condition of the conjecture from a given complete bipartite graph.

We will consider an optimal algorithm for finding a tree satisfying the condition of the conjecture from a given regular graph, and proofs and algorithms of other variations of the conjecture as further investigations.

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