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A NOTE ON LORENTZIAN METRICS OF 3-DIMENSIONAL CONFORMALLY FLAT MANIFOLDS

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Abstract

In a Lorentzian manifold the backwards triangle inequality holds. This fact is confusing to our intuition. In the study of Lorentzian manifolds, Riemannian metrics are often used. But the relation between the Lorentzian metrics and the Riemannian metrics are not clear in most cases. In this note we see that there are examples of Riemannian manifolds the Riemannian metrics of which are closely related to the Lorentzian metrics defined on them. Specifically, we introduce a Lorentzian metric on a 3-dimensional pseudo-symmetric space (M, g) of constant type. If M is, furthermore, locally conformally flat, we see that the connection of the Lorentzian metric coincides with the Riemannian connection of g and hence we have the relations of their curvatures.

Keywords: *Lorentzian metric, conformally flat, pseudo-symmetric, Ricci tensor*

1. Introduction

As is known, there is a close relation between curvature and topology of a Riemannian manifold. So, in the study of Riemannian geometry, it is a problem how curvatures (local property) determine topological structures of manifolds. Along the line, many works have been done in the last century. However, not much are known for Lorentzian manifolds.

In 1941, S. Myers proved that if the infimum of scalar curvature of a complete Riemannian manifold is positive, then the manifold is compact and the fundamental group is finite. On the other hand, E. Calabi and L. Markus proved, in 1963, that a Lorentzian spherical space form (i.e., a complete Lorentzian space of positive constant curvature) of dimension $\geqslant 3$ is noncompact and the fundamental group is finite. Hence, as contrast with Riemannian cases, it seems that unexpected phenomena happen in Lorentzian cases. The Calabi-Markus phenomena have been studied by T. Kobayashi extensively in the viewpoints of discontinuous groups.

For a constant $\varkappa$, a Riemannian manifold (M, g) whose curvature tensors R satisfies the identity

$$R(X, Y)\cdot R = \varkappa (X \wedge Y)\cdot R$$

for all tangent vector fields X and Y in M is called a *pseudo- symmetric* space of constant type, which has been investigated by O. Kowalski and many authors, mostly in 1990's. Here "$\cdot$" denotes the derivation on the algebra of *TM* induced by endomorphism of the tangent bundle *TM* and $X \wedge Y$ is the endomorphism of *TM* defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. In dimension 3, (M, g) is a pseudo-symmetric space of constant type and is not of constant curvature if and only if its principal Ricci curvatures satisfy the conditions (up to numeration)

$$\rho_1 = \rho_2 \neq \rho_3 = 2\varkappa$$

everywhere. Let $p \in M$ be an arbitrary point. Choose a sufficiently small neighborhood U of p and the smooth unit vector field E of eigenvectors of the Ricci operator corresponding to ρ_3 in U. By taking a double covering if necessary, we may assume E is a globally defined unit vector field on M. Then one may define a Lorentzian metric $\overline{g}$

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on M as follows: Let E^* be the one-form on M defined by $E^*(X) = g(E, X)$ for vector field X. We define $\overline{g} := g - 2E^* \otimes E^*$, which is a time-orientable Lorentzian metric on M. Furthermore, E becomes timelike, so the resulting Lorentzian manifold is time-orientable.

In this paper, we study the Lorentzian metric $\overline{g}$ on a 3-dimensional pseudo-symmetric space (M, g) of constant type when M is locally conformally flat, and we see that the connection of the Lorentzian metric $\overline{g}$ coincides with the Riemannian connection of g and compute their curvatures.

Our main theorem is, with the above notation, the following.

Theorem. *Let (M, g) be a 3-dimensional connected pseudo-symmetric space of constant type. Suppose that M is locally conformally flat and is not of constant curvature. If the vector field E is globally defined, then the Riemannian connection of (M, g) coincides with the Lorentzian connection of $(M, \overline{g})$.*

We always assume that manifolds are connected and smooth in this paper. In the next section, we recall basic notions and notations of pseudo-symmetric spaces of constant type. In the last section, when M is locally conformally flat, we contrast the connections of the Lorentzian metric and of the Riemannian metric g.

2. GEOMETRY OF PSEUDO-SYMMETRIC SPACES OF DIMENSION 3

In this section we recall the theory on 3-dimensional pseudo-symmetric spases developed by O. Kowalski.

O. Kowalski proved the following:

(2.1) Proposition. *Let (M, g) be a 3-dimensional pseudo-symmetric space of constant type. Then there exists a local coordinate system $(V; x, y, t)$ such that*

$$g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$$

where

$$\omega^1 = f_1(x, y, t)\,dx$$
$$\omega^2 = f_2(x, y, t)\,dy + k(x, y, t)\,dx$$
$$\omega^3 = dt + h(x, y)\,dx.$$

and $f_1 f_2 \neq 0$.

Let (M, g) be a 3-dimensional pseudo-symmetric space of constant type with constant $\varkappa$ and Q its Ricci operator. Suppose that (M, g) is not of constant curvature. Then Q has eigenvalues ρ_1, ρ_2 and ρ_3 such that $\rho_1 = \rho_2 \neq \rho_3 = 2\varkappa$. Let $(V; x, y, t)$ be the local coordinate system as in Proposition (2.1), and $\{E_1, E_2, E_3\}$ the local orthonormal frame dual to the coframe $\{\omega^1, \omega^2, \omega^3\}$. Then E_i, $i = 1, 2, 3$, are vector fields of eigenvectors of the Ricci operator Q corresponding to the eigenvalues ρ_i, respectively. We have

$$E_1 = \frac{1}{f_1 f_2}\left(f_2 \frac{\partial}{\partial x} - k\frac{\partial}{\partial y} - f_2 h\frac{\partial}{\partial t}\right)$$

$$E_2 = \frac{1}{f_2}\frac{\partial}{\partial y}$$

$$E_3 = \frac{\partial}{\partial t}$$

and

$$[E_1, E_2] = \frac{(f_1)'_y}{f_1 f_2}E_1 - \alpha E_2 + (b - c)E_3$$
$$[E_1, E_3] = aE_1 + (b + c)E_2$$
$$[E_2, E_3] = eE_2,$$

where $a = \dfrac{(f_1)'_t}{f_1}$, $b = \dfrac{1}{2f_1 f_2}(h'_y + f_2 k'_t - k(f_2)'_t)$, $e = \dfrac{(f_2)'_t}{f_2}$, $c = \dfrac{1}{2f_1 f_2}(-h'_y + f_2 k'_t - k(f_2)'_t)$, $\alpha = \dfrac{1}{f_1 f_2}((f_2)'_x - k'_y - h(f_2)'_t)$.

The Levi-Civita connection ∇ of (M, g) is given by

$$\nabla_{E_1}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{-(f_1)'_y}{f_1 f_2} & -a \\ \dfrac{(f_1)'_y}{f_1 f_2} & 0 & -c \\ a & c & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\nabla_{E_2}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & -b \\ -\alpha & 0 & -e \\ b & e & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\nabla_{E_3}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

The last identity implies that the integral curves of the vector field E_3 are geodesics, which we call the principal geodesics as in the literatures.

Recall that $\mathrm{R}(E_i, E_j)E_k = \nabla_{E_i}\nabla_{E_j}E_k - \nabla_{E_j}\nabla_{E_i}E_k - \nabla_{[E_i, E_j]}E_k$. When $i \neq j$, we denote by $K(E_i, E_j)$ the sectional curvature of two-plane section spanned by E_i and E_j:

$$K(E_i, E_j) = \frac{g(R(E_i, E_j)E_j, E_i)}{g(E_i, E_i)g(E_j, E_j) - g(E_i, E_j)^2}.$$

(2.2) Proposition. *Let (M, g) be a 3-dimensional pseudo-symmetric space of constant type with constant $\varkappa$. Then*

$$K(E_1, E_3) = -b(b+2c) - E_3(a) - a^2$$
$$K(E_2, E_3) = b^2 - E_3(e) - e^2$$
$$K(E_1, E_2) = b^2 - ea - a^2 - E_1(a) - E_2\left(\frac{(f_1)'_y}{f_1 f_2}\right) - \left(\frac{(f_1)'_y}{f_1 f_2}\right)^2.$$

(2.3) Corollary. *Let (M, g) be as in the proposition. Suppose $b = c = 0$. Then we have*

$$K(E_1, E_3) = -E_3(a) - a^2$$
$$K(E_2, E_3) = -E_3(e) - e^2,$$

and the 2-plane section spanned by E_1, E_2 is integrable.

(2.4) Remark. Note that if (M, g) is as in the Proposition (2.2),

$$\rho_1 = K(E_1, E_2) + K(E_1, E_3)$$
$$\rho_2 = K(E_1, E_2) + K(E_2, E_3)$$
$$\rho_3 = 2\varkappa = K(E_1, E_3) + K(E_2, E_3).$$
$$Sc(g) = \rho_1 + \rho_2 + 2\varkappa = 2(\rho_1 + \varkappa),$$

where $Sc(g)$ denotes the scalar curvature of (M, g).

3. (LOCALLY) CONFORMALLY FLAT SPEUDO-SYMMETRIC SPACES

3-dimensional, locally conformally flat pseudo-symmetric spaces have been studied by Hashimoto and Sekizawa. We shall quote their results for our later use in this section.

A Riemannian manifold (M, g) of dimension n is said to be *locally conformally flat* if it admits a coordinate covering $\{(V_i, \phi_i)\}$, $\phi_i : V_i \to S^n$, such that whenever $V_i \cap V_j$ is non-empty and connected, the change of coordinate map $\phi_j \circ \phi_i^{-1}$ is a conformal diffeomorphism from $\phi_i(V_i \cap V_j)$ onto $\phi_j(V_i \cap V_j)$. If $n \geqslant 3$, it follows from the Liouville theorem that $\phi_j \circ \phi_i^{-1}$ on $\phi_i(U_i \cap U_j)$ is the restriction of a Moebius transformation of S^n.

Let (M, g) be locally conformally flat. If the map $\phi_i : (V_i, g) \to (S^n, g_o)$ is conformal for each i, g is said to be compatible with the flat conformal structure, and pointwise proportional to the pull back metric $\phi_i^* g_o$. When $n = 3$, (M, g) is locally conformally flat if and only if the tensor field $Q - \frac{1}{4}Sc(g)Id$ satisfies the identity

$$(\nabla_X Q)Y - \frac{1}{4}X(Sc(g))Y = (\nabla_Y Q)X - \frac{1}{4}Y(Sc(g))X$$

for every vector fields X and Y on M. Here Id denotes the identity transformation on the tangent bundle TM.

Let (M, g) be a 3-dimensional pseudo-symmetric space of constant type with constant $\varkappa$. Suppose (M, g) is locally conformally flat. Using the above identity and recalling that $QE_i = \rho_i E_i$, $i = 1, 2, 3$, we obtain with the notations as in the previous section:

$$(\rho_1)'_t + 2(\rho_1 - 2\varkappa)a = 0, \ (\rho_1)'_y = 0$$
$$(\rho_1)'_t + 2(\rho_1 - 2\varkappa)e = 0, \ h'_y = 0$$
$$f_2(\rho_1)'_x - k(\rho_1)'_y - f_2 h(\rho_1)'_t = 0, \ b = 0 = c.$$

Hence we see

(3.1) Lemma. *Let M be a locally conformally flat 3-dimensional pseudo-symmetric space of constant type with constant $\varkappa$. Suppose that M is not of constant curvature. Then it follows that*

$$b = 0 = c \text{ and } a = e.$$

Here we note that if M is as above with constant $\varkappa \neq 0$, then the Ricci eigenvalue $\rho_1 = \rho_2$ of (M, g) different from $\rho_3 = 2\varkappa$ is not constant.

4. LORENTZIAN METRICS

Let (M, g) be a 3-dimensional pseudo-symmetric space of constant type with constant $\varkappa$. Suppose that M is not of constant curvature and is locally conformally flat. On (M, g), there is the unit vector field E_3 corresponding to the eigenvalue ρ_3 of the Ricci operator Q. Taking the double covering (if necessary), we may assume that the vector field E_3 is globally defined. Then we have the Lorentzian metric $\overline{g} := g - 2E_3^* \otimes E_3^*$ on M, where E_3^* is the one-form on M defined by $E_3^*(X) = g(E_3, X)$ for vector field X. Furthemore, E_3 becomes timelike so that the resulting Lorentzian manifold is time-orientable. For the local orthonormal frame field $\{E_1, E_2, E_3\}$ relative to g taken in the previous section, we have $\overline{g}(E_i, E_j) = g(E_i, E_j) = \delta_{ij}$ for $i = 1, 2$ and any j, but $\overline{g}(E_3, E_3) = -1$.

The Levi-Civita connection $\overline{\nabla}$ of $(M, \overline{g})$ is obtained, with the notations as in the previous section, as follows:

$$\overline{\nabla}_{E_1}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}=\begin{pmatrix}0 & \frac{-(f_1)'_y}{f_1f_2} & -a\\ \frac{(f_1)'_y}{f_1f_2} & 0 & 0\\ a & 0 & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

$$\overline{\nabla}_{E_2}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}=\begin{pmatrix}0 & a & 0\\ -a & 0 & -e\\ 0 & e & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

$$\overline{\nabla}_{E_3}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

We should note that the last identity implies the integral curves of E_3 are geodesics with respect to the Lorentzian metric $\overline{g}$ also. Consequently, we have

(4.1) Proposition. *Let $(M,\ \overline{g})$ be the Lorentzian manifold defined above. Then the integral curves of E_3 are geodesics with respect to the metric $\overline{g}$ as well as g. Moreover, if we denote by $\overline{K}(E_i, E_j)$ the sectional curvature with respect to the Lorentzian metric $\overline{g}$ of the two-plane section spanned by E_i and E_j, then*

$$\overline{K}(E_1, E_3) = E_3(a) + a^2$$
$$\overline{K}(E_2, E_3) = E_3(e) + e^2$$
$$\overline{K}(E_1, E_2) = -ea - a^2 - E_1(a) - E_2\left(\frac{(f_1)'_y}{f_1f_2}\right) - \left(\frac{(f_1)'_y}{f_1f_2}\right)^2.$$

Contrasting Propositions (2.2) and (4.1) and using Lemma 3.1, we have the following relations on curvatures of (M, g) and $(M, \overline{g})$:

(4.2) Theorem. *Let $(M,\ g)$ be a 3-dimensional pseudo-symmetric space of constant type. Suppose M is locally conformally flat and is not of constant curvature. If E_3 is globally defined, then the Riemannian connection ∇ of $(M,\ g)$ and the Lorentzian connection $\overline{\nabla}$ of $(M,\ \overline{g})$ coincide, but their curvatures satisfy the relation $K(E_i, E_3) = -\overline{K}(E_i, E_3)$ for $i = 1, 2$.*

Moreover, the plane spanned by E_1 and E_2 is parallel along the principal geodesics and integtable.

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